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FURTHER DEVELOPMENTS IN THE GLOBAL RESOLUTION OF CONVEX PROGRAMS

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Collaborative Research: Further Developments in the Global Resolution of Convex Programs with Complementary Constraints

Final Report 2011–2014

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Abstract

We have developed methods for finding globally optimal solutions to various classes of nonconvex optimization problems. We have shown that any nonconvex conic quadratically constrained quadratic program can be lifted to a convex conic optimization problem. We have shown that a complementarity approach can be used to find sparse solutions to optimization problems, with promising initial theoretical and computational results. We have investigated various relaxation approaches to several classes of problems with complementarity constraints, including linear programs with complementarity constraints, support vector regression parameter selection, bi-parametric linear complementarity constrained linear programs, quadratic programs with complementarity constraints, and nonconvex quadratically constrained quadratic programs, proving various theoretical results for each of these problems as well as demonstrating the computational effectiveness of our approaches.

1 Introduction

Six papers have been published (see the archival publications section). Four additional papers have been submitted:

[A] "On QPCCs, QCQPs and Completely Positive Programs", by L. Bai, J.E. Mitchell, and J.-S. Pang. Journal submission.

[B] "Complementarity Formulations of ℓ_0-norm Optimization Problems", by M. Feng, J.E. Mitchell, J.-S. Pang, X. Shen, and A. Wchter. September 23, 2013. Journal submission.

[C] "An Algorithm for Global Solution to Bi-Parametric Linear Complementarity Constrained Linear Programs", by Y.-C. Lee, J.-S. Pang, and J.E. Mitchell. Journal submission.

[D] "Global Resolution of the Support Vector Machine Regression Parameters Selection Problem", by Y.-C. Lee, J.-S. Pang, and J.E. Mitchell. Journal submission.

Most of these papers are available from Mitchell's webpage,

`http://www.rpi.edu/~mitchj`

Four doctoral students who were partially supported by this grant have graduated from RPI or UIUC:

- **Lijie Bai**, On convex quadratic programs with complementarity constraints, August 2013, RPI.
- **Tim Lee**, Approximations and improvements to semidefinite relaxations of optimization problems, August 2013, RPI.
- **Yu-Ching Lee**, Global solution to parametric complementarity constrained programs and applications in optimal parameter selection, August 2013, UIUC.
- **Bin Yu**, A Branch and Cut Approach to Linear Programs with Linear Complementarity Constraints, August 2011, RPI.

Further papers are in preparation, including documentation of methods for finding global optima of linear programs with complementarity constraints. Multiple talks have been given at conferences and universities.

In paper [A], we show that any quadratically constrained quadratic program is equivalent to a convex optimization problem. The result requires no assumptions on the boundedness of the feasible region or on the convexity of the quadratic constraints. The result also holds if a nonnegativity assumption on the variables is replaced by a more general convex conic constraint; for example, the result holds if the variables satisfy a second order cone constraint or if the variables satisfy a semidefiniteness constraint. The proof exploits the relationship between this class of problems and quadratic programs with complementarity constraints.

The results of [A] show that many important practical problems can be represented as convex optimization problems through the use of a single lifting. This includes rank-constrained semidefinite programs. The latter class includes problems such as factor analysis problems for finding a low-rank covariance matrix, sensor array processing, minimizing the rank of a Hankel matrix in model identification problems in system theory and signal processing, problems in systems and control, and combinatorial optimization problems.

The convex optimization problem is not easy to solve, because it is defined over a convex cone that is hard to work with. Nonetheless, the exact reformulation offers the possibility of designing new and effective algorithms for solving a broad class of optimization problems.

In paper [B], we developed a nonlinear programming formulation of the problem of minimizing the number of nonzero components in the solution of a system of linear equalities and inequalities, also known as minimizing the L0-norm. This is the problem of compressed sensing that is currently of great interest. It also arises in the search for sparse solutions to support vector machine problems in classification. Our method is slower than a linear programming approximation to the L0-norm problem, but it has the advantage that it often gives a sparser solution than the L1-norm approximation. This seems to be especially true when the system involves inequality constraints, as is the case for sparse support vector machines.

2 On QPCCs, QCQPs and Completely Positive Programs

The material in this section is drawn from our paper "On QPCCs, QCQPs and Completely Positive Programs" [2]. The quadratically constrained quadratic program, abbreviated QCQP, is a constrained optimization problem whose objective and constraint functions are all quadratic. The specification of the problem also allows linear constraints; we also allow conic convex constraints on the variables. We focus on QCQPs that may have several convex quadratic constraints but they have just one nonconvex quadratic constraint $q(x) \leq 0$, and further $q(x) \geq 0$ for any x that satisfies the linear and conic constraints:

$$
\begin{aligned}
&\underset{x \in \mathcal{K} \cap \mathcal{M}}{\text{minimize}} && f_0(x) \triangleq (\,c^0\,)^T x + \tfrac{1}{2}\, x^T Q^{\,0} x \\
&\text{subject to} && q(x) \triangleq \mathbf{h} + \mathbf{q}^T x + \tfrac{1}{2} x^T \mathbf{Q} x \,\leq\, 0 \\
&\text{and} && f_i(x) \triangleq h_i + (\,c^i\,)^T x + \tfrac{1}{2}\, x^T Q^{\,i} x \,\leq\, 0, \quad i \,=\, 1, \cdots, I,
\end{aligned}
\tag{1}
$$

where $\mathcal{M} = \{x \in \mathbb{R}^n : Ax = b\}$, where $x \in \mathcal{K} \cap \mathcal{M}$ implies $q(x) \geq 0$, and where the constraints $f_i(x) \leq 0$, $i = 1 \ldots, I$, are convex.

We denote a QCQP of this form an *nSp-QCQP*, because it has no Slater point; if $I = 0$ then we call it an *nSp0-QCQP*. We show that any QCQP can be expressed in this form with a convex objective function, generalizing a result for bounded QCQPs in [9]. Thus, the considered class of QCQPs is broad.

Quadratic programs with (linear) complementarity constraints (QPCCs) are instances of nSp0-QCQPs:

$$
\begin{aligned}
&\underset{x \triangleq (x^0, x^1, x^2)}{\text{minimize}} && c^T x + \tfrac{1}{2}\, x^T Q x \\
&\text{subject to} && Ax \,=\, b \quad \text{and} \quad \langle\, x^1,\, x^2 \rangle \leq\, 0 \\
&\text{with} && x^0 \,\in\, \mathcal{K}^0,\, x^1 \,\in\, \mathcal{K}^1,\, x^2 \,\in\, \mathcal{K}^{1*}, \\
& && \text{where } \mathcal{K}^{1*} \text{ is the dual cone to } \mathcal{K}^1.
\end{aligned}
\tag{2}
$$

Such a complementarity constraint renders the QPCC a nonconvex disjunctive program even if the objective function is convex.

The paper addresses several topics associated with the QPCC and QCQP: existence of an optimal solution to a QCQP, the formulation of a QCQP as a QPCC, the local optimality conditions of a class of quadratically constrained nonlinear programs failing constraint qualifications, and the formulation of a QCQP as a completely positive program. The paper contains a wealth of new results pertaining to the three classes of problems appearing in the title; see Figure 1. These results are all for the situation where $\mathcal{K}$ is a convex cone. The upward pointing arrows illustrate relationships where the lower problem can be regarded directly as an instance of the upper problem. The downward pointing arrows require proof and in some cases the cone is changed. Collectively, these results add significant insights to the problems. Most importantly, our study touches on a class of nonconvex programs failing the Slater constraint qualification and suggests that such problems have an underlying piecewise structure and can be converted to convex programs by a single lifting of their domain of definition.

The central problem is the nSp0-QCQP; some QCQPs are already in this form, while any other can be manipulated into this form. The manipulation also results in a convex objective function. Provided the objective is copositive on a certain subset of the recession cone of $\mathcal{K} \cap \mathcal{M}$, an nSp0-QCQP is equivalent to a convex completely positive program. This extends the recent papers [8, 9] that address the completely positive representations of binary nonconvex quadratic programs, certain types of quadratically constrained quadratic programs, and a number of other NP-hard problems. Consequently, we show that *any*

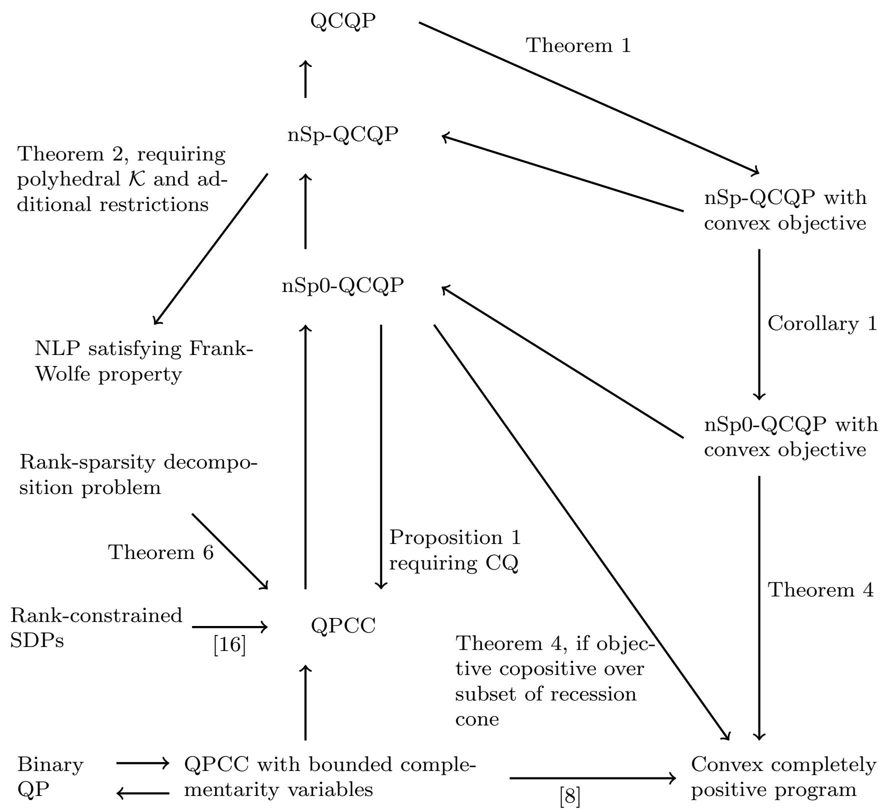

Figure 1: Diagram of results. The notation "$p \to q$" means p is a subclass of q. Theorem, Proposition, and Corollary numbers refer to the paper [2].

QCQP is equivalent to a convex completely positive program, if we first reformulate it as an nSp0-QCQP with a convex objective function. Further, a rank constrained SDP is also equivalent to a convex completely positive program, since it is equivalent to a QPCC over the semidefinite cone, with a linear objective function. Similarly, the rank-sparsity decomposition problem is equivalent to a convex program; in many applications, it is desired to split a matrix into a low rank part and a sparse part, which can help with inference [12].

The existence of an optimal solution to a convex QCQP over the nonnegative orthant is fully resolved via the classical Frank-Wolfe theorem [5], which states that such a minimization program, if feasible, has an optimal solution if and only if the objective function of the program is bounded below on the feasible set. The situation with a nonconvex QCQP is rather different; the sharpest Frank-Wolfe type existence results for a feasible QCQP with a nonconvex (quadratic) objective are obtained in [25]. We extend their work as follows:

Theorem 1 *Assume the cone $\mathcal{K}$ is polyhedral. Then the FW attainment result holds for the nSp-QCQP (1) if either*

1. $I \leq 1$ *or*

2. $q_0(x)$ is a quasiconvex function on $\mathcal{K} \cap \mathcal{M}$.

3 Complementarity Formulations of ℓ_0-norm Optimization Problems

The material in this section is drawn from our paper "Complementarity Formulations of ℓ_0-norm Optimization Problems" [18].

Denoted by $\| \bullet \|_0$, the so-called ℓ_0-norm of a vector is the number of nonzero components of the vector. In recent years, there has been an increased interest in solving optimization problems that minimize or restrict the number of nonzero elements of the solution vector [4, 7, 11, 13, 14, 15, 30, 32]. A simple example of such a problem is that of finding a solution to a system of linear inequalities with the least ℓ_0-norm:

$$\begin{aligned} &\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad \| x \|_0 \\ &\text{subject to} \quad Ax \geq b \quad \text{and} \quad Cx = d, \end{aligned} \tag{3}$$

where $A \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{k \times n}$, $b \in \mathbb{R}^m$ and $d \in \mathbb{R}^k$ are given matrices and vectors, respectively. Since this problem is NP-hard, one popular solution approach replaces the nonconvex discontinuous ℓ_0-norm in (3) by the convex continuous ℓ_1-norm, leading to a linear program:

$$\begin{aligned} &\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad \| x \|_1 \\ &\text{subject to} \quad Ax \geq b \quad \text{and} \quad Cx = d. \end{aligned} \tag{4}$$

Theoretical result are known that provide sufficient conditions under which an optimal solution to (4) is also optimal to (3) [10, 17, 19, 33]. Yet these results are of limited practical value as the conditions can not easily be verified or guaranteed for specific realizations of (3); thus in general, optimal solutions to (4) provide suboptimal solutions to (3).

It is our contention that, from a practical perspective, improved solutions to (3) can be obtained by reformulating the ℓ_0-norm in terms of complementarity constraints [24]. This leads to a linear program with linear complementarity constraints (LPCC) which can be solved with specialized algorithms that do not depend on the feasibility and/or boundedness of the constraints [20, 21]. In the event that bounds are known on the solutions of the problem, the LPCC can be further reformulated as a mixed-integer linear program (MILP). However, the solution of this MILP is usually too time-consuming for large instances.

As an alternative to the MILP approach, the LPCC can be expressed directly as a smooth continuous nonlinear program (NLP). It is the main purpose of this research to examine the quality of solutions computed by standard NLP solvers applied to these smooth reformulations of the ℓ_0-norm. There are two properties of the NLP reformulations that make them difficult to solve. First, the NLPs are highly nonconvex, and, consequently, the solutions returned by the NLP solvers depend strongly on the starting point, because the NLP methods are typically only able to find local minimizers or Karush-Kuhn-Tucker

(KKT) points, instead of global minimizers. Secondly, the NLPs are not well-posed in the sense that they do not satisfy the assumptions that are made usually for the convergence analysis of standard NLP algorithms, such as constraint qualifications. Nevertheless, our numerical results show that these methods often generate high-quality solutions for the ℓ_0-norm problem (3).

Together, the ℓ_0-norm and its complementarity formulation allow a host of minimization problems involving the count of variables to be cast as disjunctive programs with complementarity constraints. A general NLP model of this kind is as follows: for two finite index sets $\mathcal{E}$ and $\mathcal{I}$,

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) + \gamma \| \, x \, \|_0 \\ \text{subject to} & c_i(x) \, = \, 0, \quad i \, \in \, \mathcal{E} \\ \text{and} & c_i(x) \, \leq \, 0, \quad i \, \in \, \mathcal{I}, \end{array} \tag{5}$$

where $\gamma > 0$ is a prescribed scalar and the objective function f and the constraint functions c_i are all continuously differentiable. A distinguished feature of this problem is that its objective function is discontinuous, in fact lower semicontinuous; as such, it attains its minimum over any compact set, but in general the existence/attainment of an optimal solution is not immediately clear. Among other things, the reformulation presented below offers a constructive venue for establishing the solvability of the problem, under reasonable conditions.

We can derive an equivalent formulation as a complementarity problem:

$$\begin{array}{ll} \underset{x,\, x^{\pm},\, \xi}{\text{minimize}} & f(x) + \boldsymbol{\gamma}^T \, (\, \mathbf{1}_n - \xi \,) \\ \text{subject to} & c_i(x) \, = \, 0, \quad i \, \in \, \mathcal{E} \\ & c_i(x) \, \leq \, 0, \quad i \, \in \, \mathcal{I} \\ & x = x^+ - x^- \\ & 0 \, \leq \, \xi \, \perp \, x^+ + x^- \, \geq \, 0 \\ & 0 \, \leq \, x^+ \, \perp \, x^- \, \geq \, 0 \\ \text{and} & \xi \, \leq \, \mathbf{1}_n, \end{array} \tag{6}$$

where we have used an arbitrary given positive vector $\boldsymbol{\gamma}$ instead of a scalar γ-multiple of the vector of ones. This time, the statement involves nonlinear objective and constraint functions, giving rise to a Mathematical Program with Complementarity Constraints (MPCC). It is not difficult to deduce that if x is an optimal solution of (5), then by letting $x^{\pm} \triangleq \max(0, \pm x)$ and

$$\xi_j \, \triangleq \left\{ \begin{array}{ll} 0 & \text{if } x_j \, \neq \, 0 \\ 1 & \text{if } x_j \, = \, 0 \end{array} \right\} \quad j \, = \, 1, \cdots, n, \tag{7}$$

the resulting triple $(x^{\pm}, \xi)$ is an optimal solution of (6) with objective value equal to $\|x\|_0$. Conversely, if $(x^{\pm}, \xi)$ is an optimal solution of (6), then $x \triangleq x^+ - x^-$ is an optimal solution

of (5) with the same objective value as the optimal objective value of (6). The definition (7) provides a central connection between (5) and its "pieces".

We may associate with (5) the following smooth NLP with an auxiliary variable ξ, which we call a half complementarity formulation:

$$
\begin{array}{ll}
\underset{x,\,\xi}{\text{minimize}} & f(x) + \boldsymbol{\gamma}^T\,(\,\mathbf{1}_n - \xi\,) \\
\text{subject to} & c_i(x) \,=\, 0, \quad i \,\in\, \mathcal{E} \\
& c_i(x) \,\leq\, 0, \quad i \,\in\, \mathcal{I} \\
& 0 \,\leq\, \xi \,\leq\, \mathbf{1}_n \quad \text{and} \quad \xi \circ x \,=\, 0.
\end{array}
\tag{8}
$$

The paper contains theorems relating solutions to (5) with KKT points for (6) and (8). We also examine relaxed formulations of (6) and (8), and show interesting convergence properties as the relaxation is tightened. In particular, we can show for certain choices of f and c that any subsequence of local minimizers converges to a solution with strong theoretical properties.

We give one graph to give a flavor of the computational results in the paper. This is for a problem of the form

$$
\begin{array}{ll}
\underset{x\,\in\,\mathbb{R}^n}{\text{minimize}} & \|\,x\,\|_0 \\
\text{subject to} & Ax \,\geq\, b \quad \text{and} \quad -M\mathbf{1}_n \,\leq\, x \,\leq\, M\mathbf{1}_n,
\end{array}
\tag{9}
$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $M > 0$. The test instances were generated using AMPL's internal random number generator, where the elements of A and b are independent uniform random variables between -1 and 1. Our numerical experiments compare the performance of different NLP optimization codes when they are applied to the different NLP reformulations. Because these problems are nonconvex, we also explore the effect of different starting points.

Our numerical study suggests that standard NLP codes are able to generate solutions that can be significantly better than those obtained by convex approximations of the NP-hard problem. This is somewhat remarkable because the NLP formulations are highly nonconvex and the usual constraint qualifications, such as MFCQ, do not hold. Our numerical experiments did not identify a clear winner among the different reformulations of the ℓ_0-norm minimization problems. Similarly, while some NLP codes tended to produce better results than others, it is not clear which specific features of the algorithms or their implementations are responsible for finding good solutions. We point out that each software implementation includes enhancements, such as tricks to handle numerical problems due to round-off error or heuristics that are often not included in the mathematical description in scientific papers. Because the NLP reformulations of the ℓ_0-problems are somewhat ill-posed, these enhancement are likely to be crucial for the solver's performance. Once the relevant ingredients of the reformulation and optimization method have been identified, it might be possible to design specialized NLP-based algorithms that are tailored to the task of finding sparse solutions efficiently.

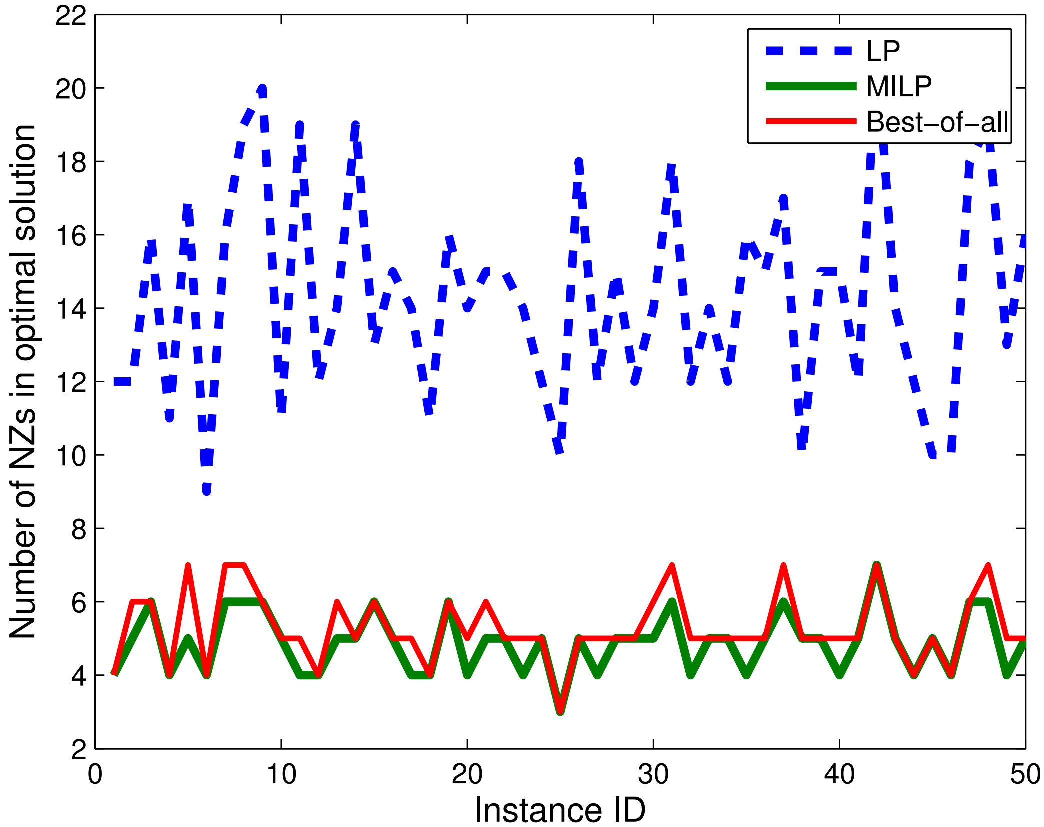

Figure 2: The sparsest solutions for the 50 random problems using different NLP solvers

Finally, the numerical study in this paper has been performed using randomly generated model problems. Future efforts will explore the suitability of the proposed approach for ℓ_0-norm optimization problems arising in particular applications areas including compressive sensing [11, 15], basis pursuit [7, 14], machine learning [13, 26], and genome-wide association studies [32].

4 Algorithms for Global Resolution of Mathematical Programs with Complementarity Constraints

4.1 General methods for convex MPCCs

In the paper "Obtaining tighter relaxations of mathematical programs with complementarity constraints" [28], we investigated cutting plane approaches to MPCCs. In an effort to find a global optimum, it is often useful to examine the relaxation obtained by omitting the complementarity constraints. We discuss various methods to tighten the relaxation by exploiting complementarity, with the aim of constructing better approximations to the convex hull of the set of feasible solutions to the MPCC, and hence better lower bounds on the optimal value of the MPCC. Better lower bounds can be useful in branching schemes to find a globally optimal solution. Different types of linear constraints are constructed, including cuts based on bounds on the variables and various types of disjunctive cuts. Novel convex quadratic constraints are introduced, with a derivation that is particularly useful when the number of design variables is not too large. A lifting process is specialized to MPCCs. Semidefinite programming constraints are also discussed. All these constraints are typically applicable to any convex program with complementarity constraints. Computational results for linear programs with complementarity constraints (LPCCs) are included, comparing the benefit of the various constraints on the value of the relaxation, and showing that the constraints can dramatically speed up the solution of the LPCC.

In the paper "Using quadratic convex reformulation to tighten the convex relaxation of a quadratic program with complementarity constraints" [3], we looked at a preprocessing technique based on semidefinite programming. Quadratic Convex Reformulation (QCR) is a technique that has been proposed for binary and mixed integer quadratic programs. In this paper, we extend the QCR method to convex quadratic programs with linear complementarity constraints (QPCCs). Due to the complementarity relationship between the nonnegative variables y and w, a term $y^T Dw$ can be added to the QPCC objective function, where D is a nonnegative diagonal matrix chosen to maintain the convexity of the objective function and the global resolution of the QPCC. Following the QCR method, the products of linear equality constraints can also be used to perturb the QPCC objective function, with the goal that the new QP relaxation provides a tighter lower bound. By solving a semidefinite program, an equivalent QPCC can be obtained whose QP relaxation is as tight as possible. In addition, we extend the QCR to a general quadratically constrained quadratic program (QCQP), of which the QPCC is a special example. Computational tests on QPCCs are presented.

In the paper "On Convex Quadratic Programs with Linear Complementarity Constraints" [1], we looked at splitting the feasible region into two sets, so as to best exploit an integer programming solver and a logical Benders decomposition approach. The paper shows that the global resolution of a general convex quadratic program with complementarity constraints (QPCC), possibly infeasible or unbounded, can be accomplished in finite time. The method constructs a minmax mixed integer formulation by introducing finitely many binary variables, one for each complementarity constraint. Based on the primal-dual relationship of a pair of convex quadratic programs and on a logical Benders scheme, an extreme ray/point generation procedure is developed, which relies on valid satisfiability constraints for the integer program. To improve this scheme, we propose a two-stage approach wherein the first stage solves the mixed integer quadratic program with pre-set upper bounds on the complementarity variables, and the second stage solves the program outside this bounded region by the Benders scheme. We report computational results with our method. We also investigate the addition of a penalty term yDw to the objective function, where y and w are the complementary variables and D is a nonnegative diagonal matrix. The matrix D can be chosen effectively by solving a semidefinite program, ensuring that the objective function remains convex. The addition of the penalty term can often reduce the overall runtime by at least 50%. We report preliminary computational testing on a QP relaxation method which can be used to obtain better lower bounds from infeasible points; this method could be incorporated into a branching scheme. By combining the penalty method and the QP relaxation method, more than 90% of the gap can be closed for some QPCC problems.

4.2 Bi-parametric linear complementarity constrained linear programs

In the paper "An Algorithm for Global Solution to Bi-Parametric Linear Complementarity Constrained Linear Programs" [22] we consider the following problem. Given variables $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ and parameters $q \in \mathbb{R}^m$, $N \in \mathbb{R}^{m\times n}$ and $M \in \mathbb{R}^{m\times m}$, the complementarity constraints

$$0 \le y \perp w := q + Nx + My$$

can be represented equivalently using the nonconvex quadratic constraint

$$q^T y + y^T My + y^T Nx \le 0$$

together with nonnegativity restrictions on y and w. In this paper, we focused on problems with $n = 2$ and with M positive semidefinite, a situation that arises often in practice (see [23] for example). We developed a domain partitioning scheme for the problem, branching on x, and proved finite convergence. We obtained promising computational results on our test problems.

4.3 Parameter selection in support vector machine regression

The support vector machine (SVM) regression problem has two design parameters: C to control the regularization and ε to control the width of a regression tube. Previously, we

have developed an LPCC formulation of the problem of choosing the two parameters [6]. In the paper "Global Resolution of the Support Vector Machine Regression Parameters Selection Problem" [23], we apply the algorithm of [22] to solve the parameter selection problem. Through the use of the algorithm, we are able to find globally optimal values of C and ε.

5 Other Results

In the paper "Convex Quadratic Relaxations of Nonconvex Quadratically Constrained Quadratic Programs" [29], we extend the technique introduced in [28] for MPCCs to nonconvex QCQPs. Nonconvex quadratic constraints can be linearized to obtain relaxations in a well-understood manner. We propose to tighten the relaxation by using second order cone constraints, resulting in a convex quadratic relaxation. Our quadratic approximation to the bilinear term is compared to the linear McCormick bounds. The second order cone constraints are based on linear combinations of pairs of variables. With good bounds on these linear combinations, the resulting constraints strengthen the McCormick bounds. Computational results are given, which indicate that the convex quadratic relaxation can dramatically improve the solution times for some problems.

We develop a homotopy approach to finding the solution to an LPCC in the paper "A Globally Convergent Probability-One Homotopy for Linear Programs with Linear Complementarity Constraints" [31]. A solution of the standard formulation of a linear program with linear complementarity constraints (LPCC) does not satisfy a constraint qualification. A family of relaxations of an LPCC, associated with a probability-one homotopy map, proposed here is shown to have several desirable properties. The homotopy map is nonlinear, replacing all the constraints with nonlinear relaxations of NCP functions. Under mild existence and rank assumptions, (1) the LPCC relaxations RLPCC(λ) have a solution for $0 \leq \lambda \leq 1$, (2) RLPCC(1) is equivalent to LPCC, (3) the Kuhn-Tucker constraint qualification is satisfied at every local or global solution of RLPCC(λ) for almost all $0 \leq \lambda \leq 1$, (4) a point is a local solution of RLPCC(1) (and LPCC) if and only if it is a Kuhn-Tucker point for RLPCC(1), and (5) a homotopy algorithm can find a Kuhn-Tucker point for RLPCC(1). Since the homotopy map is a globally convergent probability-one homotopy, robust and efficient numerical algorithms exist to find solutions of RLPCC(1). Numerical results are included for some small problems.

In the paper "Rebalancing an Investment Portfolio in the Presence of Convex Transaction Costs including Market Impact Costs" [27], we look at a class of QCQPs arising in financial optimization. The inclusion of transaction costs is an essential element of any realistic portfolio optimization. In this paper, we extend the standard portfolio problem to consider convex transaction costs that are incurred to rebalance an investment portfolio. Market impact costs measure the effect on the price of a security that result from an effort to buy or sell the security, and they can constitute a large part of the total transaction costs. The loss to a portfolio from market impact costs is typically modeled with a convex function that can usually be expressed using second order cone constraints. The Markowitz framework of

mean-variance efficiency is used. In order to properly represent the variance of the resulting portfolio, we suggest rescaling by the funds available after paying the transaction costs. This results in a fractional programming problem, which can be reformulated as an equivalent convex program of size comparable to the model without transaction costs. An optimal solution to the convex program can always be found that does not discard assets. The results of the paper extend the classical Markowitz model to the case of convex transaction costs in a natural manner with limited computational cost.

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